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A formula for the sectional geometric genus of quasi-polarized manifolds by using intersection numbers

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Abstract

Let (X,L) be a quasi-polarized manifold of $\dim X = n$ defined over the complex number field. In a previous paper, for every integer i with $0 \leqslant i \leqslant n$, we defined the ith sectional geometric genus $g_i(X,L)$ of (X,L). In this paper, we give a formula for $g_i(X,L)$ by using intersection numbers.

MSC: Primary: 14C20; secondary: 14C17, 14C40, 05A

0. Introduction

Let X be a projective variety of $\dim X = n$ defined over the complex number field, and let L be an ample (resp. a nef and big) line bundle on X. Then (X,L) is called a *polarized* (*resp. quasi-polarized*) *variety*. If X is smooth, then we say that (X,L) is a polarized (resp. quasi-polarized) *manifold*. In order to study polarized varieties, it is important to use various invariants of (X,L). There are the following three invariants of (X,L) which are well-known.

(1) The degree L^n.
(2) The sectional genus $g(L)$.
(3) The Δ-genus $\Delta(L)$.

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By using these invariants, many authors studied polarized varieties. In particular, Ionescu classified polarized manifolds by their degree under the assumption that L is very ample with $L^n \leqslant 8$ [7–9], and Fujita classified polarized varieties by their Δ-genus and their sectional genus [2].

In [3], in order to study polarized varieties more deeply, the author introduced the notion of the ith sectional geometric genus $g_i(X,L)$ of (X,L) for every integer i with $0 \leqslant i \leqslant n$. This is a generalization of the degree and the sectional genus of (X,L). Namely $g_0(X,L) = L^n$ and $g_1(X,L) = g(L)$.

Here, we recall the reason why this invariant is called the ith sectional geometric genus. Let (X,L) be a polarized manifold of dimension $n \geqslant 2$ with $\mathrm{Bs}\,|L| = \emptyset$, where $\mathrm{Bs}\,|L|$ is the base locus of the complete linear system $|L|$. We put $X_0 := X$. Let i be an integer with $0 \leqslant i \leqslant n-1$, and let X_{n-i} be the transversal intersection of $n-i$ general members of $|L|$. In this case X_{n-i} is a smooth projective variety of dimension i. Then, we can prove that $g_i(X,L) = h^i(\mathcal{O}_{X_{n-i}})$, that is, $g_i(X,L)$ is the geometric genus of X_{n-i}.

Hence, we can expect that $g_i(X,L)$ has properties analogous to that of the geometric genus of i-dimensional varieties.

An expression for $g_i(X,L)$ by using intersection numbers is very useful. For example, in [4, Corollary 2.3], for $i=2$ we obtained a formula for $g_2(X,L)$ by using intersection numbers (see also (2.2.A) below). By using the adjunction theory and a lower bound for $c_2(X)L^{n-2}$, the formula for $g_2(X,L)$ by using intersection numbers enabled us to get a lower bound for $g_2(X,L)$. In particular, if $\kappa(X) \geqslant 0$, then we were able to show that $g_2(X,L) \geqslant h^1(\mathcal{O}_X)$ for $n \geqslant 4$ (see [4, Corollary 3.5.2]). Therefore it is important to get a formula by using intersection numbers.

In this paper, for any quasi-polarized manifolds (X,L) and every integer i with $0 \leqslant i \leqslant n-1$ we give a formula for $g_i(X,L)$ by using intersection numbers in general (see Theorem 2.1). We note that in this formula, we use the Stirling number of the second kind (see Definition 1.5 below).

The contents of this paper are as follows. In Section 1, we give some results which are used in Section 2. In particular, we give a method for calculating $S(n,n-i)$ (see Remark 1.7). In Section 2, we prove the main theorem (Theorem 2.1), and by using this, we calculate $g_2(X,L)$, $g_3(X,L)$, and $g_4(X,L)$ (see (2.2.A), (2.2.B), and (2.2.C)).

1. Preliminaries

Definition 1.1 (Fukuma [3, Definition 2.1]). Let (X,L) be a quasi-polarized variety of $\dim X = n$, and let $\chi(tL)$ be the Euler–Poincaré characteristic of tL. Here we put

$$\chi(tL) = \sum_{j=0}^{n} \chi_j(X,L)\,\frac{t^{[j]}}{j!},$$

where $t^{[j]} = t(t+1)\cdots(t+j-1)$ for $j \geqslant 1$ and $t^{[0]} = 1$. Then for every integer i with $0 \leqslant i \leqslant n$ the ith *sectional geometric genus* $g_i(X,L)$ of (X,L) is defined by the following formula:

$$g_i(X,L) = (-1)^i(\chi_{n-i}(X,L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X).$$

Remark 1.1.1. (1) Since $\chi_{n-i}(X,L) \in \mathbb{Z}$, $g_i(X,L)$ is an integer by definition.

(2) If $i = 0$ (resp. $i = 1$), then $g_0(X,L)$ (resp. $g_1(X,L)$) is equal to the degree (resp. the sectional genus) of (X,L).

(3) If $i = n$, then $g_n(X,L) = h^n(\mathcal{O}_X)$, and $g_n(X,L)$ is independent of L.

Theorem 1.2. *Let (X,L) be a quasi-polarized variety of* $\dim X = n$. *Let i be an integer with $0 \leqslant i \leqslant n-1$. Then*

$$g_i(X,L) = \sum_{j=0}^{n-i-1}(-1)^{n-j}\binom{n-i}{j}\chi(-(n-i-j)L) + \sum_{k=0}^{n-i}(-1)^{n-i-k}h^{n-k}(\mathcal{O}_X).$$

Proof. By the following equality, which was proved in [3, Theorem 2.2], we obtain the assertion. For every integer p with $0 \leqslant p \leqslant n$,

$$\chi_p(X,L) = \sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}\chi(-(p-k)L). \qquad \square$$

Proposition 1.3. *Let $x_0 = 1$ and let x_i be an indeterminate of weight i for every integer i with $i \geqslant 1$. For any non-negative integer k, there exist unique polynomials of weight k, $T_k \in \mathbb{Q}[x_1,\dots,x_k]$, such that the following properties hold*:

(1) $T_0 = 1$.

(2) *For any formal power series $\sum_{i=0}^{\infty} x_i t^i$, we put*

$$\mathrm{td}_t\left(\sum_{i=0}^{\infty} x_i t^i\right) = \sum_{k=0}^{\infty} T_k(x_1,\dots,x_k)t^k,$$

where t is an indeterminate.

If

$$\sum_{i=0}^{\infty} x_i t^i = \left(\sum_{i=0}^{\infty} y_i t^i\right)\left(\sum_{i=0}^{\infty} z_i t^i\right),$$

then

$$\mathrm{td}_t\left(\sum_{i=0}^{\infty} x_i t^i\right) = \left(\mathrm{td}_t\left(\sum_{i=0}^{\infty} y_i t^i\right)\right)\left(\mathrm{td}_t\left(\sum_{i=0}^{\infty} z_i t^i\right)\right).$$

(3) *For the linear expression $1 + xt$,*

$$\mathrm{td}_t(1+xt) = \frac{xt}{1-\exp(-xt)}.$$

Proof. See [6, Chapter 1, Section 1]. □

Definition 1.4. (1) Polynomials $T_k \in \mathbb{Q}[x_1,\ldots,x_k]$ in Proposition 1.3 is called the *Todd polynomial of weight* k.

(2) Let X be a smooth projective variety and let $\mathscr{F}$ be a vector bundle on X. Let $c_t(\mathscr{F}) = \sum_{i\geqslant 0} c_i(\mathscr{F})t^i$ be the Chern polynomial of $\mathscr{F}$. We put

$$\mathrm{td}_t(\mathscr{F}) = \mathrm{td}\left(\sum_{i\geqslant 0} c_i(\mathscr{F})t^i\right) = \sum_{k=0}^{\infty} T_k(c_1(\mathscr{F}),\ldots,c_k(\mathscr{F}))t^k,$$

where t is an indeterminate. Furthermore, we put

$$\mathrm{td}(\mathscr{F}) := \sum_{k=0}^{\infty} T_k(c_1(\mathscr{F}),\ldots,c_k(\mathscr{F})).$$

Then $\mathrm{td}(\mathscr{F})$ (resp. $T_k(c_1(\mathscr{F}),\ldots,c_k(\mathscr{F}))$) is called the *Todd class of* $\mathscr{F}$ (resp. *Todd polynomial of weight* k *of* $\mathscr{F}$). In particular, if $\mathscr{F}$ is the tangent bundle T_X of X, then we put $T_k(X) := T_k(c_1(T_X),\ldots,c_k(T_X))$.

Remark 1.4.1. For a formal power series $\sum_{k=0}^{\infty} p_k t^k$, the Todd polynomial of weight 1 (resp. 2 and 3) $T_1(p_1)$ (resp. $T_2(p_1,p_2)$ and $T_3(p_1,p_2,p_3)$) are the following:

$$T_1(p_1) = \tfrac{1}{2}\,p_1,$$

$$T_2(p_1,p_2) = \tfrac{1}{12}\,(p_2 + p_1^2),$$

$$T_3(p_1,p_2,p_3) = \tfrac{1}{24}\,p_1p_2,$$

$$T_4(p_1,p_2,p_3,p_4) = \tfrac{1}{720}\,(-p_1^4 + 4p_1^2p_2 + 3p_2^2 + p_1p_3 - p_4).$$

(See [6, Chapter 1].)

Definition 1.5 (Berge [1, Chapter 1, Section 10]). Let m and n be non-negative integers. Then $S(n,m)$ denotes the number of partitions of a set n objects into m classes. We call this number $S(n,m)$ the *Stirling number of the second kind with the type* (n,m). For convenience, we put $S(0,0)=1$.

Remark 1.5.1. (1) Let $S(p,q)$ be the Stirling number of the second kind with the type (p,q). Then the following hold:

(1.1) $S(p,q)=0$ for $p<q$,
(1.2) $S(p,0)=0$ for $p\geqslant 1$,
(1.3) $S(p,1)=S(p,p)=1$,
(1.4) $S(p,q)=S(p-1,q-1)+qS(p-1,q)$ for $1\leqslant q\leqslant p$.

(2) By [5, Lemma 3.A.5], we obtain the following:

(2.1) $S(n,n-1)=n(n-1)/2$ for $n\geqslant 2$,
(2.2) $S(n,n-2)=n(n-1)(n-2)(3n-5)/24$ for $n\geqslant 3$,
(2.3) $S(n,n-3)=n(n-1)(n-2)^2(n-3)^2/48$ for $n\geqslant 4$.

Proposition 1.6. *Let $S(n, n-i)$ be the Stirling number of the second kind with the type $(n, n-i)$, where n is an indeterminate and i is an integer with $1 \leqslant i \leqslant n$. Then $S(n, n-i)$ is a polynomial in n with rational coefficients whose degree is $2i$ and is divisible by $n(n-1)\cdots(n-i)$.*

Proof. We prove this by induction on i.

If $i = 1$, then this is true by Remark 1.5.1(2.1).

Assume that the assertion is true if $i = k-1$. We consider the case where $i = k$. By Remark 1.5.1(1.4) we obtain that

$$S(n, n-k) = \sum_{j=k+1}^{n} (j-k)S(j-1, j-k).$$

We put $F(j;k) := S(j-1, j-k)$. When we consider $S(j-1, j-k)$ as a polynomial in j, we use $F(j;k)$.

By assumption $F(j;k)$ is divisible by $(j-1)(j-2)\cdots(j-k)$. Hence

$$\begin{aligned} S(n, n-k) &= \sum_{j=k+1}^{n} (j-k)S(j-1, j-k) \\ &= \sum_{j=1}^{n} (j-k)F(j;k). \end{aligned}$$

Here we put

$$G(n;k) := \sum_{j=1}^{n} (j-k)F(j;k).$$

Since $F(j;k)$ is a polynomial in j with rational coefficients whose degree is $2(k-1)$ by induction hypothesis, $G(n;k)$ is a polynomial in n with rational coefficients whose degree is $2k$, and since $F(j;k)$ is divisible by $(j-1)\cdots(j-k)$, we get that $G(n;k)=0$ for every n with $1 \leqslant n \leqslant k$. Therefore we obtain that $G(n;k)$ is divisible by $(n-1)(n-2)\cdots(n-k)$.

Here we note that

$$\begin{aligned} G(n;k) &= \sum_{j=1}^{n} \left(\sum_{t=0}^{2k-1} p_t j^t \right) \\ &= \sum_{t=0}^{2k-1} p_t \left(\sum_{j=1}^{n} j^t \right) \end{aligned}$$

for $p_t \in \mathbb{Q}$. Since $\sum_{j=1}^{n} j^t$ is divisible by n for every integer t with $t \geqslant 0$, $G(n;k)$ is divisible by n.

Therefore we get the assertion. □

Remark 1.7. (1) By Remark 1.5.1(1.4), we get that

$$S(n,n-i)-S(n-1,n-1-i)=(n-i)S(n-1,n-i).$$

Then

$$\frac{1}{(n-1)\cdots(n-i)}\left(S(n,n-i)-S(n-1,n-1-i)\right)$$
$$=\frac{(n-i)}{(n-1)\cdots(n-i)}S(n-1,n-i). \tag{1.7.1}$$

First we calculate the left-hand side of (1.7.1). By Proposition 1.6, $S(n,n-i)n^{-1}(n-1)^{-1}\cdots(n-i)^{-1}$ is a polynomial in n of degree $i-1$ and we can put

$$S(n,n-i)=n(n-1)\cdots(n-i)\left(\sum_{j=0}^{i-1}a_{i,j}n^j\right),$$

where $a_{i,j}\in\mathbb{Q}$. Then

$$\begin{aligned}\frac{S(n,n-i)-S(n-1,n-1-i)}{(n-1)\cdots(n-i)}&=n\sum_{j=0}^{i-1}a_{i,j}n^j-(n-i-1)\sum_{j=0}^{i-1}a_{i,j}(n-1)^j\\&=\sum_{j=0}^{i-1}a_{i,j}n^{j+1}-(n-i-1)\sum_{j=0}^{i-1}a_{i,j}(n-1)^j.\end{aligned}$$

On the other hand

$$\begin{aligned}(n-1-i)\sum_{j=0}^{i-1}a_{i,j}(n-1)^j&=\sum_{j=0}^{i-1}\left(\sum_{k=j}^{i-1}a_{i,k}(-1)^{k-j}\binom{k}{j}\right)n^{j+1}\\&\quad-\sum_{j=0}^{i-1}\left(\sum_{k=j}^{i-1}a_{i,k}(-1)^{k-j}\binom{k}{j}(i+1)\right)n^j.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{j=0}^{i-1}a_{i,j}n^{j+1}-(n-i-1)\sum_{j=0}^{i-1}a_{i,j}(n-1)^j\\&\quad=\sum_{j=0}^{i-1}\left(a_{i,j}-\sum_{k=j}^{i-1}a_{i,k}(-1)^{k-j}\binom{k}{j}\right)n^{j+1}\\&\qquad+\sum_{j=0}^{i-1}\left(\sum_{k=j}^{i-1}a_{i,k}(-1)^{k-j}\binom{k}{j}(i+1)\right)n^j\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{i-1} \left(a_{i,j-1} - \sum_{k=j-1}^{i-1} a_{i,k}(-1)^{k-j+1} \binom{k}{j-1} \right) n^j \\
&\quad + \sum_{j=1}^{i-1} \left(\sum_{k=j}^{i-1} a_{i,k}(-1)^{k-j} \binom{k}{j} (i+1) \right) n^j + \sum_{k=0}^{i-1} a_{i,k}(-1)^k (i+1) \\
&= \sum_{j=1}^{i-1} \left(a_{i,j-1} + \sum_{k=j-1}^{i-1} a_{i,k}(-1)^{k-j} \binom{k}{j-1} \right. \\
&\quad \left. + \sum_{k=j}^{i-1} a_{i,k}(-1)^{k-j} \binom{k}{j} (i+1) \right) n^j + \sum_{k=0}^{i-1} a_{i,k}(-1)^k (i+1) \\
&= \sum_{j=1}^{i-1} \left(\sum_{k=j}^{i-1} a_{i,k}(-1)^{k-j} \binom{k}{j-1} + \sum_{k=j}^{i-1} a_{i,k}(-1)^{k-j} \binom{k}{j} (i+1) \right) n^j \\
&\quad + \sum_{k=0}^{i-1} a_{i,k}(-1)^k (i+1). \qquad (1.7.2)
\end{aligned}
$$

Next we calculate the right-hand side of equality (1.7.1). Then

$$
\begin{aligned}
&\frac{n-i}{(n-1)\cdots(n-i)} S(n-1, n-i) \\
&\quad = (n-i) \sum_{j=0}^{i-2} a_{i-1,j}(n-1)^j \\
&\quad = (n-i) \sum_{j=0}^{i-2} \left(\sum_{k=j}^{i-2} a_{i-1,k}(-1)^{k-j} \binom{k}{j} \right) n^j \\
&\quad = \sum_{j=1}^{i-1} \left(\sum_{k=j-1}^{i-2} a_{i-1,k}(-1)^{k-j+1} \binom{k}{j-1} \right) n^j \\
&\qquad - \sum_{j=0}^{i-2} \left(\sum_{k=j}^{i-2} a_{i-1,k}(-1)^{k-j} \binom{k}{j} i \right) n^j \\
&\quad = \sum_{j=1}^{i-2} \left(\sum_{k=j-1}^{i-2} a_{i-1,k}(-1)^{k-j+1} \binom{k}{j-1} \right.
\end{aligned}
$$

$$+\sum_{k=j}^{i-2} a_{i-1,k}(-1)^{k-j+1}\binom{k}{j}i\Bigg)n^j$$

$$+a_{i-1,i-2}n^{i-1}-\sum_{k=0}^{i-2}a_{i-1,k}(-1)^k i. \tag{1.7.3}$$

By comparing Eqs. (1.7.2) and (1.7.3), we obtain

$$2ia_{i,i-1}=a_{i-1,i-2}, \tag{1.7.4}$$

$$\begin{aligned}&\sum_{k=j}^{i-1}a_{i,k}(-1)^k\left(\binom{k}{j-1}+\binom{k}{j}(i+1)\right)\\ &\quad=\sum_{k=j-1}^{i-2}a_{i-1,k}(-1)^{k+1}\binom{k}{j-1}+\sum_{k=j}^{i-2}a_{i-1,k}(-1)^{k+1}\binom{k}{j}i\\ &\quad\text{for } 1\leqslant j\leqslant i-2,\end{aligned} \tag{1.7.5}$$

$$\sum_{k=0}^{i-1}a_{i,k}(-1)^k(i+1)=-\sum_{k=0}^{i-2}a_{i-1,k}(-1)^k i. \tag{1.7.6}$$

By using (1.7.4), (1.7.5) and (1.7.6), we obtain that $(a_{i,i-1},\dots,a_{i,0})$ if we know $(a_{i-1,i-2},\dots,a_{i-1,0})$.

(2) For example, by using the above argument, we calculate $S(n,n-4)$. By Remark 1.5.1(2.3) $(a_{3,2},a_{3,1},a_{3,0})=(1/48,-5/48,1/8)$. Therefore by (1.7.4), (1.7.5) and (1.7.6) we obtain that

$$8a_{4,3}=\tfrac{1}{48},$$

$$18a_{4,3}-7a_{4,2}=\tfrac{11}{48},$$

$$16a_{4,3}-11a_{4,2}+6a_{4,1}=\tfrac{5}{6},$$

$$5(a_{4,3}-a_{4,2}+a_{4,1}-a_{4,0})=1.$$

Then

$$a_{4,3}=\tfrac{1}{384},\quad a_{4,2}=-\tfrac{5}{192},\quad a_{4,1}=\tfrac{97}{1152},\quad a_{4,0}=-\tfrac{251}{2880}.$$

Hence

$$S(n,n-4)=\tfrac{1}{5760}n(n-1)(n-2)(n-3)(n-4)(15n^3-150n^2+485n-502).$$

By the same argument as above, we can calculate $S(n,n-i)$ for $i\geqslant 5$.

(3) By using Eqs. (1.7.4), (1.7.5) and (1.7.6), for every integer i with $i \geqslant 2$ we get an $i \times i$ matrix $P_i = (p_{j,k})$ and an $i \times (i-1)$ matrix $Q_i = (q_{l,m})$ such that

$$P_i \begin{pmatrix} a_{i,i-1} \\ a_{i,i-2} \\ \vdots \\ a_{i,0} \end{pmatrix} = Q_i \begin{pmatrix} a_{i-1,i-2} \\ a_{i-1,i-3} \\ \vdots \\ a_{i-1,0} \end{pmatrix} \tag{1.7.7}$$

and $p_{j,k}$ and $q_{l,m}$ satisfy the following:

$$\begin{aligned} &p_{j,j} = (-1)^{j-1}(2i-(j-1)) \quad \text{if } 1 \leqslant j \leqslant i, \\ &p_{i,k} = (-1)^{k-1}(i+1) \quad \text{if } 1 \leqslant k \leqslant i, \\ &p_{j,k} = 0 \quad \text{if } j < k, \\ &p_{j-1,k} + p_{j,k} = -p_{j-1,k-1} \quad \text{if } 2 \leqslant k \leqslant j-1 \leqslant i-1 \end{aligned} \tag{1.7.8}$$

and

$$\begin{aligned} &q_{l,l} = (-1)^{l-1} \quad \text{if } 1 \leqslant l \leqslant i-1, \\ &q_{i,m} = (-1)^{m-1} i \quad \text{if } 1 \leqslant m \leqslant i-1, \\ &q_{l,m} = 0 \quad \text{if } l < m, \\ &q_{l-1,m} + q_{l,m} = -q_{l-1,m-1} \quad \text{if } 2 \leqslant m \leqslant l-1 \leqslant i-1. \end{aligned} \tag{1.7.9}$$

(Here we note that by the above relations (1.7.8) and (1.7.9), we can determine P_i and Q_i.)

Since $\det P_i \neq 0$, there exists the inverse matrix P_i^{-1} for every integer i with $i \geqslant 2$. Hence by using (1.7.7), we obtain that

$$\begin{aligned} \begin{pmatrix} a_{i,i-1} \\ a_{i,i-2} \\ \vdots \\ a_{i,0} \end{pmatrix} &= P_i^{-1} Q_i \begin{pmatrix} a_{i-1,i-2} \\ a_{i-1,i-3} \\ \vdots \\ a_{i-1,0} \end{pmatrix} \\ &= P_i^{-1} Q_i P_{i-1}^{-1} P_{i-1} \begin{pmatrix} a_{i-1,i-2} \\ a_{i-1,i-3} \\ \vdots \\ a_{i-1,0} \end{pmatrix} \end{aligned}$$

$$= P_i^{-1} Q_i P_{i-1}^{-1} Q_{i-1} \begin{pmatrix} a_{i-2,i-3} \\ a_{i-2,i-4} \\ \vdots \\ a_{i-2,0} \end{pmatrix}$$

$$= \cdots$$

$$= P_i^{-1} Q_i P_{i-1}^{-1} Q_{i-1} \cdots P_2^{-1} Q_2 (a_{1,0})$$

$$= P_i^{-1} Q_i P_{i-1}^{-1} Q_{i-1} \cdots P_2^{-1} Q_2 \left(\tfrac{1}{2}\right).$$

(Here we note that $a_{1,0} = 1/2$ by Remark 1.5.1(2.1).)

Proposition 1.8. *Let n and m be non-negative integers. Then*

$$S(n,m) = \frac{1}{m!} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^n.$$

Proof. See [1, Chapter 3, Section 1, p. 79]. □

2. Main theorem

First we prove the following main theorem.

Theorem 2.1. *Let (X,L) be a quasi-polarized manifold of $\dim X = n \geqslant 2$. For every integer i with $0 \leqslant i \leqslant n-1$,*

$$g_i(X,L) = \sum_{l=0}^{n-1} (-1)^l \frac{(n-i)!}{(n-l)!} S(n-l, n-i) T_l(X) L^{n-l} + (-1)^{i+1} \left(\chi(\mathcal{O}_X) - \sum_{k=0}^{n-i} (-1)^{n-k} h^{n-k}(\mathcal{O}_X) \right).$$

(*Here $T_l(X)$ denotes the Todd polynomial of weight l of the tangent bundle T_X of X* (*see Definition* 1.4(2)), *and $S(n-l, n-i)$ denotes the Stirling number of the second kind of type $(n-l, n-i)$.*)

Proof. By Theorem 1.2, we obtain that

$$g_i(X,L) = \sum_{j=0}^{n-i-1} (-1)^{n-j} \binom{n-i}{j} \chi(-(n-i-j)L) + \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(\mathcal{O}_X).$$

Next we calculate $\chi(-(n-i-j)L)$. By the Hirzebruch–Riemann–Roch theorem ([6, Chapter 4]), we obtain that

$$\begin{aligned}\chi(-tL) &= \int_X \mathrm{ch}(-tL)\,\mathrm{td}(\mathrm{T}_X)\\ &= \sum_{l=0}^{n} \frac{(-t)^{n-l}}{(n-l)!} T_l(X)L^{n-l}.\end{aligned}$$

(Here $\mathrm{ch}(-tL)$ denotes the Chern character of $-tL$.) Hence

$$\begin{aligned}&g_i(X,L)\\ &\quad= \sum_{j=0}^{n-i-1} (-1)^{n-j} \binom{n-i}{j} \left\{ \sum_{l=0}^{n} \frac{(-(n-i-j))^{n-l}}{(n-l)!} T_l(X)L^{n-l} \right\}\\ &\qquad+ \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(\mathcal{O}_X)\\ &\quad= \sum_{l=0}^{n} \left\{ \sum_{j=0}^{n-i-1} (-1)^{j} \binom{n-i}{j} (n-i-j)^{n-l} \right\} \frac{(-1)^l}{(n-l)!} T_l(X)L^{n-l}\\ &\qquad+ \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(\mathcal{O}_X).\end{aligned}$$

By Proposition 1.8, we obtain that

$$\begin{aligned}&\sum_{j=0}^{n-i-1} (-1)^{j} \binom{n-i}{j} (n-i-j)^{n-l}\\ &\quad= \sum_{j=0}^{n-i-1} (-1)^{j} \binom{n-i}{n-i-j} (n-i-j)^{n-l}\\ &\quad= \sum_{j=0}^{n-i} (-1)^{(n-i)-(n-i-j)} \binom{n-i}{n-i-j} (n-i-j)^{n-l} - (-1)^{n-i} \binom{n-i}{0} 0^{n-l}\\ &\quad= \begin{cases} (n-i)!S(n-l,n-i) & \text{if } l \neq n\\ (n-i)!S(0,n-i) + (-1)^{n-i+1} & \text{if } l = n \end{cases}\\ &\quad= \begin{cases} (n-i)!S(n-l,n-i) & \text{if } l \neq n\\ (-1)^{n-i+1} & \text{if } l = n. \end{cases}\end{aligned}$$

Therefore

$$
\begin{aligned}
g_i(X,L)
&= \sum_{l=0}^{n} \left\{ \sum_{j=0}^{n-i-1} (-1)^j \binom{n-i}{j} (n-i-j)^{n-l} \right\} \frac{(-1)^l}{(n-l)!} T_l(X)L^{n-l} \\
&\quad + \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(\mathcal{O}_X) \\
&= \sum_{l=0}^{n-1} \{(n-i)!S(n-l,n-i)\} \frac{(-1)^l}{(n-l)!} T_l(X)L^{n-l} \\
&\quad + (-1)^{n-i+1}(-1)^n T_n(X) + \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(\mathcal{O}_X) \\
&= \sum_{l=0}^{n-1} (-1)^l \frac{(n-i)!}{(n-l)!} S(n-l,n-i)\, T_l(X)L^{n-l} \\
&\quad + (-1)^{i+1} \left(\chi(\mathcal{O}_X) - \sum_{k=0}^{n-i} (-1)^{n-k} h^{n-k}(\mathcal{O}_X) \right)
\end{aligned}
$$

and we get the assertion. □

By using Theorem 2.1, Remarks 1.4.1, 1.5.1 and 1.7(2), we calculate $g_2(X,L)$, $g_3(X,L)$ and $g_4(X,L)$.

The case where $i=2$. Then

$$
\begin{aligned}
g_2(X,L) &= \sum_{l=0}^{n-1} (-1)^l \frac{(n-2)!}{(n-l)!} S(n-l,n-2) T_l(X)L^{n-l} \\
&\quad + (-1)^3 \left(\chi(\mathcal{O}_X) - \sum_{k=0}^{n-2} (-1)^{n-k} h^{n-k}(\mathcal{O}_X) \right) \\
&= \frac{(n-2)(3n-5)}{24} L^n + \frac{n-2}{4} K_X L^{n-1} \\
&\quad + \frac{1}{12} (K_X^2 + c_2(X)) L^{n-2} - 1 + h^1(\mathcal{O}_X). \qquad (2.2.\mathrm{A})
\end{aligned}
$$

This has already been obtained in [4, Corollary 2.3].

The case where $i=3$. Then

$$\begin{aligned}
g_3(X,L) &= \sum_{l=0}^{n-1}(-1)^l \frac{(n-3)!}{(n-l)!} S(n-l,n-3)T_l(X)L^{n-l} \\
&\quad + (-1)^4\left(\chi(\mathcal{O}_X) - \sum_{k=0}^{n-3}(-1)^{n-k}h^{n-k}(\mathcal{O}_X)\right) \\
&= \frac{(n-2)(n-3)^2}{48}L^n + \frac{(n-3)(3n-8)}{48}K_XL^{n-1} \\
&\quad + \frac{n-3}{24}(K_X^2+c_2(X))L^{n-2} + \frac{1}{24}K_Xc_2(X)L^{n-3} \\
&\quad + 1 - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X). \qquad (2.2.\mathrm{B})
\end{aligned}$$

The case where $i=4$. Then

$$\begin{aligned}
g_4(X,L) &= \sum_{l=0}^{n-1}(-1)^l \frac{(n-4)!}{(n-l)!} S(n-l,n-4)T_l(X)L^{n-l} \\
&\quad + (-1)^5\left(\chi(\mathcal{O}_X) - \sum_{k=0}^{n-4}(-1)^{n-k}h^{n-k}(\mathcal{O}_X)\right) \\
&= \tfrac{1}{5760}(n-4)(15n^3-150n^2+485n-502)L^n \\
&\quad + \tfrac{1}{96}(n-3)(n-4)^2K_XL^{n-1} + \tfrac{1}{288}(n-4)(3n-11) \\
&\quad \times (K_X^2+c_2(X))L^{n-2} + \tfrac{1}{48}(n-4)K_Xc_2(X)L^{n-3} \\
&\quad + \tfrac{1}{720}(-K_X^4+4K_X^2c_2(X)+3c_2(X)^2-K_Xc_3(X)-c_4(X))L^{n-4} \\
&\quad - 1 + h^1(\mathcal{O}_X) - h^2(\mathcal{O}_X) + h^3(\mathcal{O}_X). \qquad (2.2.\mathrm{C})
\end{aligned}$$

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